**Jacques Gélinas** <jacquesgg00@gmail.com>

**To:**LinXiao

Sun, Mar 13, 2022 at 6:02 AM

URL: <http://archive.ymsc.tsinghua.edu.cn/pacm_download/117/5376-11511_2007_Article_BF02565336.pdf>

@article{Polya:1926,  
  author  = "George {P{\'o}lya}",  
  title   = "Bemerkung {\"u}ber die {I}ntegraldarstellung der {R}iemannschen {$\xi$}-{F}unktion",  
  journal = Acta Mathematica,  
  volume  = 48,  
  year    = 1926,  
  pages   = "305--317",  
  note    = "Reprinted as item 93 in \cite{Polya:1974}"  
}  
@book{Polya:1974,  
  author    = "George {P{\'o}lya}",  
  title     = "Collected papers II",  
  publisher = "The Clarendon Press",  
  address   = "Cambridge Mass.",  
  year      = 1974  
}

Unfortunately, this is in German but you should be able to read some equations since you know the subject well.

0) In 1913, Landau mentionned the problem of finding the distribution of zeros for modified Xi\*(t) functions obtained by truncating the infinite series phi(t) to a finite sum, and Polya gave an answer in 1926. (Polya does not use the Landau convention but uses xi(t) like Riemann insted of our Xi(t) = xi(1/2+it))

1) Hillfsatz 1 (at the end) shows that the cosine transform of a fast decreasing function like phi(t) does not change sign near positive infinity on the real line if that function is not even. So he concludes that the Jensen kernel phi(t) must be even, and truncating it destroys that symmetry so the modified Xi\*(t) will have a finite number of real zeros, and an infinite number of complex zeros (from the Hadamard theory all these entire transforms are even of order one and must have an infinite number of zeros).

2) So Polya chooses the even modified function in (5) instead of (4), and he shows that it looks much like like the Xi(t) function AND has only real roots (!!), The proof uses first the Hermite-Biehler theorem generalized to entire functions in Hillfsatz 2 (adapting the proof of Laguerre for the original H-B polynomial theorem). Thus, from equation (8), it remains to study the simpler function in (7), ''die einige schöne, einfache Eigenschaften aufweist'' which has nice, simple properties.

3) The rest of the article studies the entire function from (7). Polya uses all the tools that he knows on this problem. In particular, he shows that this entire function has only simple and pure imaginary zeros inside its critical strip (p 313)..

4. Now I can read German, but the mathematics used here are not so transparent for me, notably p 314-315. The equation (18) is the important one:

if A(z)F(z) -1 = B(z), and B(z) is smaller than 1, then obviously F(z) is not zero, so the problem is to bound B(z).

5. At the end, another modified Xi\*\*(t) function is indicated (found during the correction of the preprint) and it mimics better still the real Xi(t).

Maybe the techniques used by Polya can inspire you ?

Jacques Gélinas